CONTROL OF A MIMO SYSTEM IN DISCRETE TIME

1. Introduction

The aim of this lesson is to control a hydraulic system which has two inputs and two outputs, that is to say, a nonlinear MIMO (multiple-input and multiple-output) system. First, we design a discrete-time controller that can be based either on classical or on modern control theory. After that, we open the HTML page and configure the parameters of the plant. Next, we calculate and introduce the parameters of the controller and run the simulation. Finally, we can see the evolution of the controlled system and check if it is the desired behavior.

This page provides the following configurations to control this system:

- PID controller
- State feedback control
- Observer and state feedback
- Integral controller, observer and state feedback.

The controller is obtained using the linearized system, but the page permits visualizing the evolution of the original nonlinear system, with the purpose of checking if the designed controller is able to control it.

In summary, we will design different controllers for a MIMO system, and see its behavior in an HTML page.

2. Description of the system

This plant consists of a valve and pump that distributes fluid to two tanks, as show in Figure 1. The valve position determines how the flow from the pump is divided between both tanks.

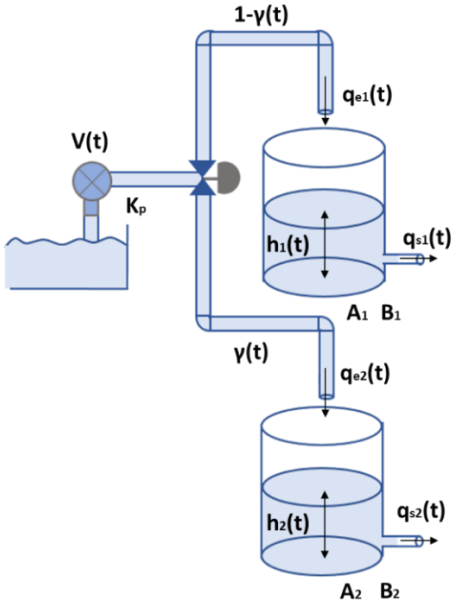


Figure 1. MIMO physical system.

The parameters of the process are the cross section of each tank (A_1 and A_2), the cross section of their output pipes (B_1 and B_2), the constant gain of the pump (K_p) and the acceleration of gravity (g).

On the other hand, the variables are the voltage applied to the pump $V(t)$, the opening of the valve $\gamma(t) \in [0,1]$, the input flow to tank 1 $q_{e1}(t)$, the output flow from the tank 1 $q_{s1}(t)$, the input flow to tank 2 $q_{e2}(t)$, the output flow from the tank 2 $q_{s2}(t)$ and the height of fluid in each tank $h_1(t)$ and $h_2(t)$.

Once the parameters and variables are defined, the equations of this system are:

$$q_{e1}(t) = K_p \cdot V(t) \cdot \big(1 - \gamma(t)\big) \quad (1) \qquad q_{e2}(t) = K_p \cdot V(t) \cdot \gamma(t) \quad (2)$$

$$q_{s1}(t) = B_1 \cdot \sqrt{2 \cdot g \cdot h_1(t)} \quad (3) \qquad q_{s2}(t) = B_2 \cdot \sqrt{2 \cdot g \cdot h_2(t)} \quad (4)$$

$$q_{e1}(t) - q_{s1}(t) = A_1 \cdot \frac{dh_1(t)}{dt} \quad (5) \qquad q_{e2}(t) - q_{s2}(t) = A_2 \cdot \frac{dh_2(t)}{dt} \quad (6)$$

The process inputs are the voltage to the pump and the opening of the valve, and the outputs are the fluid heights in both tanks.

Exercise 1:

Linearize the system about the working point defined by $V_0 = 6\,V$ and $\gamma_0 = 0.4$. Obtain a state-space representation of the linearized system using the heights of fluid as state variables.

$x_1 = h_1(t)$	$x_2 = h_2(t)$
$u_1 = V(t)$	$u_2 = \gamma(t)$

Consider the next parameters:

Parameters	Values	Units
Kp	3.75	$\mathrm{m}^3/(\mathrm{V} \cdot \mathrm{s})$
B1	1.5	m^2
B1	1.5	m^2
A1	20	m^2
A2	20	m^2
g	9.8	m^2/s

3. How to use HTML page

When we open the HTML page, the first thing that appears is a menu with three options: parameters, classic control and state-space control, as shown in Figure 2.

The first step is to define all the parameters of the plant as well as the operating point. For it, we have to click on the parameters option. In this window (Figure 3), we can select the kind of equations to use to simulate the controlled system: nonlinear or linearized system.

Then, we return to previous menu (Figure 2) to choose the kind of controller (this will be explained in subsection 3.1). Once the controller has been configured, we can start the simulation and we will visualize the window shown in Figure 4. In the right half of this window, the system evolution can be observed through drawing elements.

Also, there is a slider and a parsed field, which permit changing the SPD (Steps Per Display) value. This parameter determines how many steps the calculations advance before refreshing the simulations. With SPD at 5 the simulation will be shown as in real time. We can increase its value with the purpose that the system could advance multiple steps before the visualization is redrawn.

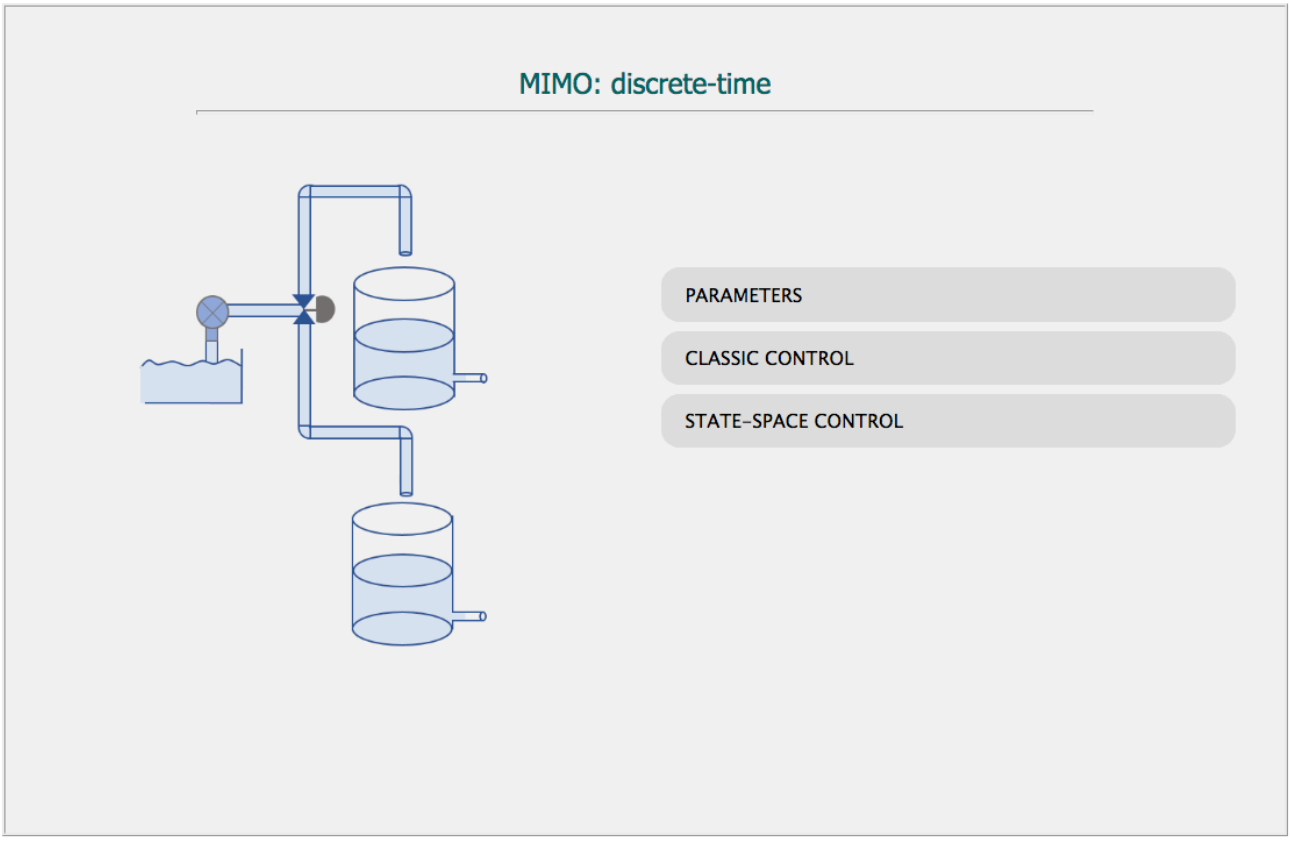


Figure 2. Menu window.

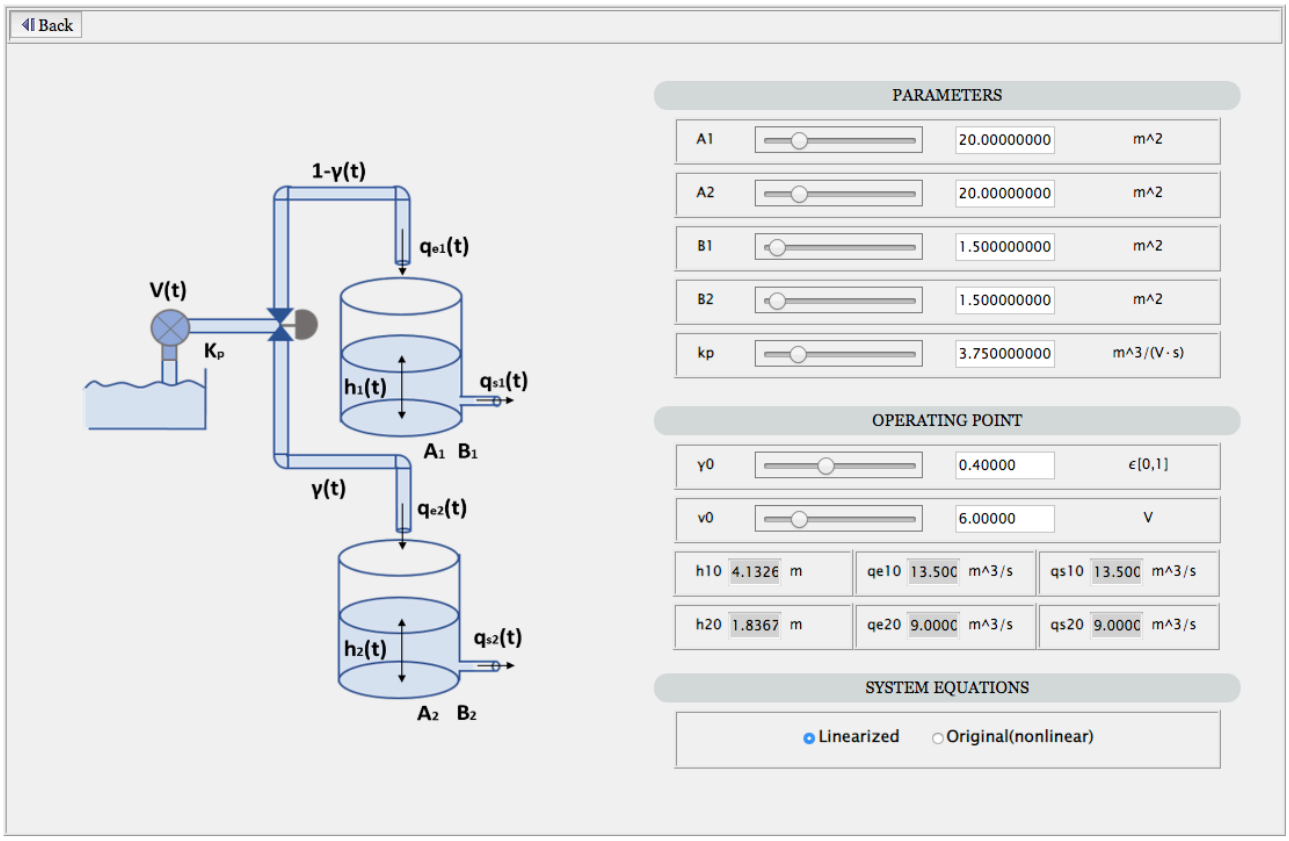


Figure 3. Parameters window.

On the left half, there are four buttons to choose the variable which will be visualized, a graphic that shows the evolution of the selected variable and a set of buttons:

- Button play: the simulation is stopped when the page is opened, so if we click on this button the simulation will start.
- Button pause: this button pauses the simulation when it is running.
- Button reset: we have to click on this button if we want the value of each variable to be their initial value. The back button does it as well, but also returns to the previous menu.

On the bottom of Figure 4, we can introduce the reference value of each height and the actual value of them is also shown in two text fields as the simulation advances.

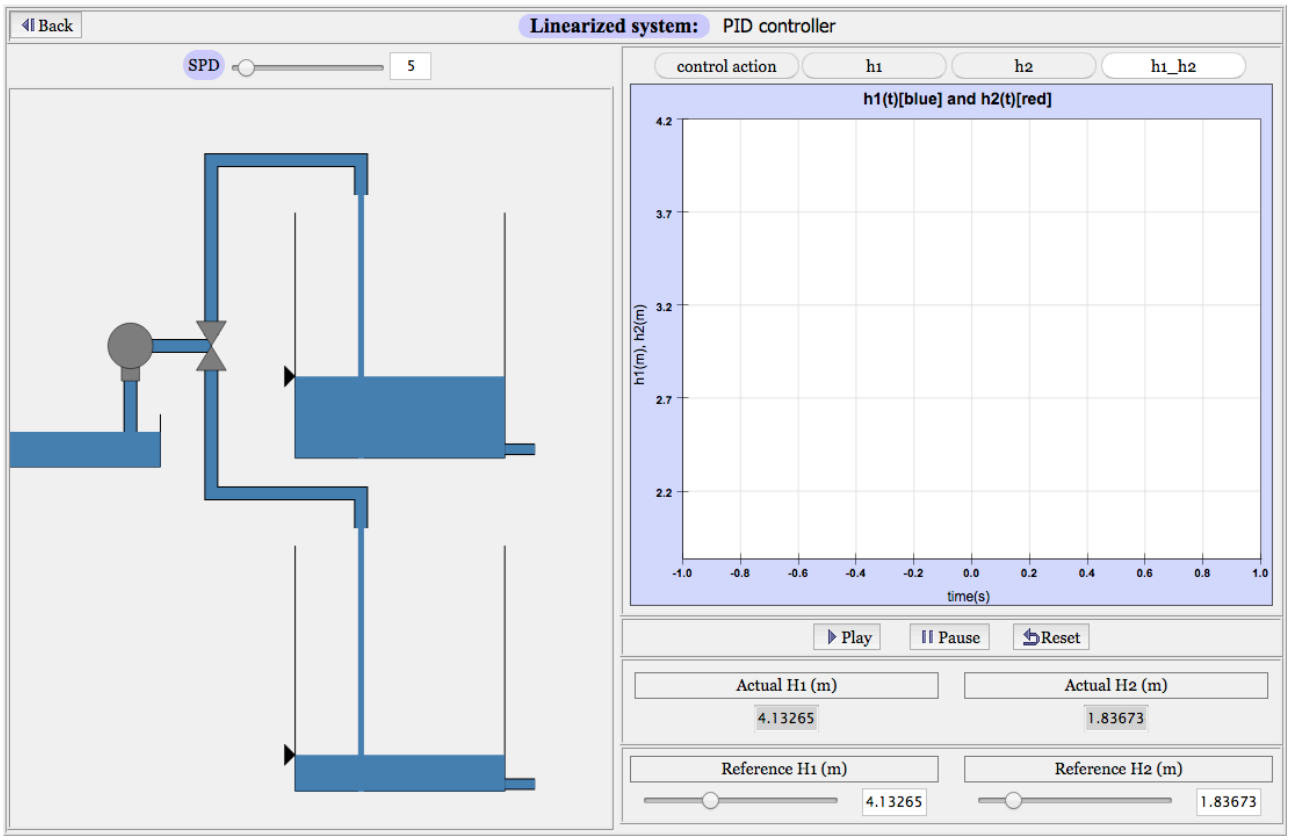


Figure 4. Simulation window.

3.1 Types of control systems

The system can be controlled using several kinds of control systems based either on classical or on modern control theory. The next subsections show all the available controllers and some examples.

a) PID CONTROLLER

The first one is a PID controller, whose parameters will be calculated from the linearized system and the required specifications. Figure 5 shows a block diagram of the controlled system using this controller, where ZOH represents a zero-order hold and SYSTEM is the plant.

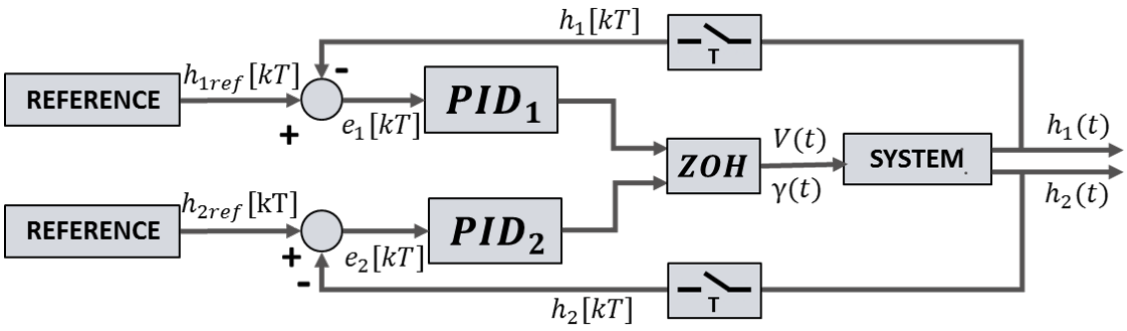


Figure 5. Control schema: PID controller.

The gain of the controller can be tuned using either any analytic method, or experimentally. In the second case, the gain can be set following the next steps, to control this MIMO hydraulic system.

1) Firstly, the proportional constant (Kp) is increased until the desired rise time is obtained (before the overshoot be excessive).
2) Next, to obtain a smaller overshoot, the derivative constant (Kd) must be increased.
3) Finally, we have to increase the value of the integral constant if there is a steady state error, until setting it.

These values, as well as the sample time Ts (seconds), must be introduced in the window of this controller (Figure 6), where there is also a saturation block corresponding to the voltage of the pump. For instance, supposing that the motor works correctly when voltage value is between ±10V. In other words, the voltage control action signal does not take values greater than 10V nor less than -10V. So, we must activate the upper and lower saturation, clicking the selector. After that the maximum and minimum value must be introduced.

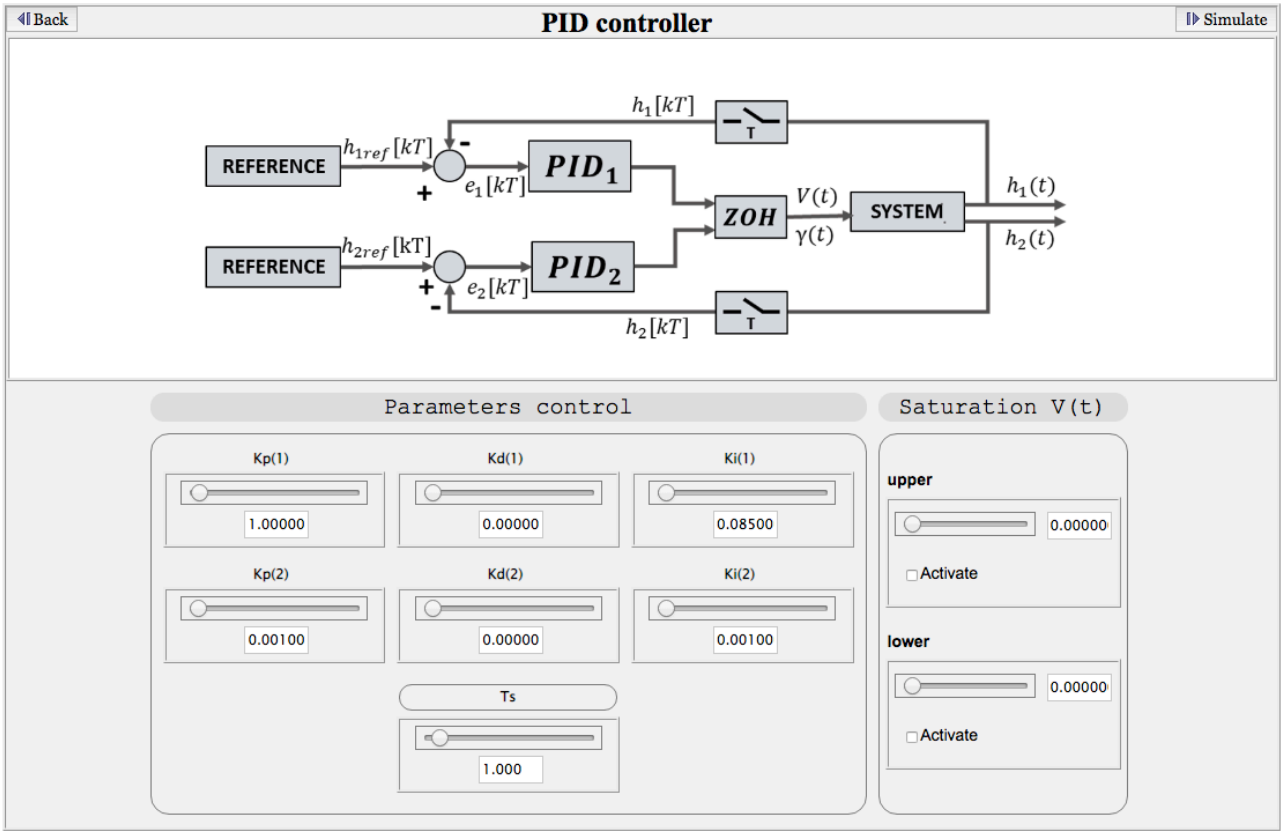


Figure 6. PID controller window.

Exercise 2:

Open the HTML page and choose classic control. Using the default values (Figure 6) as parameters of the PID controller, click on the simulate button.
Then, introduce the following reference inputs. Finally, check if the outputs track their respective reference values.

Reference H1	5.5 m
Reference H2	2.5 m

Does the control system work correctly when used to control the original nonlinear system?

b) STATE FEEDBACK CONTROL

The first type of control based on modern theory is the state feedback control, which permits placing the poles so that the controlled system meets the specifications.

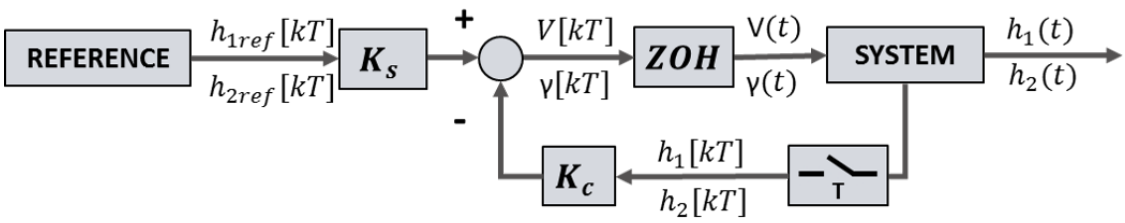


Figure 7. Control schema: State Feedback control.

To control the system using feedback control, it is necessary to obtain the state feedback matrix Kc, which defines the dynamics of the system and the input gain Ks, which adjusts the steady state. Figure 7 shows a block diagram of the controlled system using this controller. The next lines present a complete example of designing this kind of controller:

Once the linearized equations are obtained, the state-space representation is:

$$\begin{bmatrix}\dot{h}_1\\ \dot{h}_2\end{bmatrix} = \begin{bmatrix}-\frac{B_1}{A_1}\sqrt{\frac{g}{2\cdot h_{10}}} & 0\\ 0 & -\frac{B_2}{A_2}\sqrt{\frac{g}{2\cdot h_{20}}}\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix} + \begin{bmatrix}K_p\cdot\frac{1-\gamma_0}{A_1} & -K_p\cdot\frac{V_0}{A_1}\\ K_p\cdot\frac{\gamma_0}{A_2} & K_p\cdot\frac{V_0}{A_2}\end{bmatrix}\begin{bmatrix}V\\ \gamma\end{bmatrix}$$

$$\begin{bmatrix}h_1\\ h_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix}$$

The control of the system is discrete time, so we have to discretize it. A sample time equal to 1 second is considered and a zero-order hold is assumed.

$$G = \begin{bmatrix}0.9216 & 0\\ 0 & 0.8847\end{bmatrix} \qquad H = \begin{bmatrix}0.1080 & -1.0803\\ 0.0706 & 1.0588\end{bmatrix}$$

To design the feedback matrix, the state space representation must be expressed in controllable canonic form. The transformation matrix is obtained using the controllability matrix Q. In general:

$$Q = [h_1 h_2 \dots h_m \quad Gh_1 Gh_2 \dots Gh_m \quad \dots \quad G^{n-1}h_1 G^{n-1}h_2 \dots G^{n-1}h_m]$$

where h_i is the i-th column in the matrix H. This system has two state variables ($n = 2$) and two inputs ($m = 2$), so the controllability matrix is:

$$Q = [h_1 h_2 \quad Gh_1 Gh_2] = \begin{bmatrix}0.1080 & -1.0803 & 0.09953 & -0.9956\\ 0.0706 & 1.0588 & 0.06246 & 0.9367\end{bmatrix}$$

The rank of the matrix Q is equal to two, so the system is controllable. The next step is to select n linearly independent columns. The selected columns will determine how many state variable are controlled with each input. We select:

$$QR = [h_1 h_2] = \begin{bmatrix}0.1080 & -1.0803\\ 0.0706 & 1.0588\end{bmatrix}$$

then one state variable is controlled with u_1 and the other with u_2. The columns have to be ordered according to its association to each input (first those associated to u_1, second…). In the equation, the column h_i is associated to the input i, and n_i is the number of columns in QR associated to u_i.

$$L = [h_1 \quad G\cdot h_1 \quad \dots \quad G^{n_1-1}\cdot h_1 \quad \dots \quad h_d \quad G\cdot h_d \quad \dots \quad G^{n_d-1}\cdot h_d]$$

In this case, all inputs take part in the feedback, so $d = 2$ and each one has a column associated ($n_1 = 1$ and $n_2 = 1$).

$$L = \begin{bmatrix}0.1080 & -1.0803\\ 0.0706 & 1.0588\end{bmatrix}$$

The next step is to calculate the matrix inverse of L. Then, we extract the final row of each block to obtain the inverse of the transformation matrix T^{-1}. In general:

$$L^{-1} = \begin{bmatrix} e_1^T \\ \vdots \\ e_{n1}^T \\ e_{n1+1}^T \\ \vdots \\ e_{n1+n_2}^T \\ \vdots \\ \vdots \\ e_{n1+n2+\cdots+n_d}^T \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} e_{n_1}^T \\ e_{n_1}^T \cdot G \\ \vdots \\ e_{n1}^T \cdot G^{n_1-1} \\ e_{n1+n_2}^T \\ \vdots \\ e_{n1+n_2}^T \cdot G^{n_2-1} \\ \vdots \\ e_{n1+n2+\cdots+n_d}^T \\ \vdots \\ e_{n1+n2+\cdots+n_d}^T \cdot G^{n_d-1} \end{bmatrix}$$

In this case:

$$L^{-1} = \begin{bmatrix} e_1^T \\ e_2^T \end{bmatrix} = \begin{bmatrix} 5.5545 & 5.6673 \\ -0.3704 & 0.5666 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} e_1^T \\ e_2^T \end{bmatrix} = \begin{bmatrix} 5.5545 & 5.6673 \\ -0.3704 & 0.5666 \end{bmatrix}$$

Once the transformation matrix is obtained, the matrices of the system can be expressed in controllable canonic form.

$$\tilde{G} = T^{-1} \cdot G \cdot T = \begin{bmatrix} 0.9068 & -0.2214 \\ -0.1476e-02 & 0.8995 \end{bmatrix}$$

$$\tilde{H} = T^{-1} \cdot H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, we are going to design the feedback matrix. We stablish these specifications: the controlled system must be critically damped with a settling time approximately equal to 53 seconds. In continuous time the poles are placed at $s_{1,2} = -0.09$, so; in the z plane:

$$z_{1,2} = e^{s \cdot T_s} = e^{-0.09 \cdot 1} = 0.9139$$

In order to meet the specifications, the desired characteristic polynomial of the system must be:

$$P(z) = z^2 - 1.8278z + 0.83521$$

The feedback matrix is calculated according to the next equation:

$$\tilde{G}_R = \tilde{G} - \tilde{H} \cdot \tilde{K}_C$$

where, the matrix $\tilde{G}_R$ is:

$$\tilde{G}_R = \begin{bmatrix} z_1 & 1 \\ 0 & z_2 \end{bmatrix} = \begin{bmatrix} 0.9139 & 1 \\ 0 & 0.9139 \end{bmatrix}$$

Therefore, the previous equation is:

$$\begin{bmatrix} 0.9139 & 1 \\ 0 & 0.9139 \end{bmatrix} = \begin{bmatrix} 0.9068 & -0.2214 \\ -0.1476e-02 & 0.8995 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{k}_{11} & \tilde{k}_{12} \\ \tilde{k}_{21} & \tilde{k}_{22} \end{bmatrix}$$

$$\begin{bmatrix} 0.9139 & 1 \\ 0 & 0.9139 \end{bmatrix} = \begin{bmatrix} 0.9068 - \tilde{k}_{11} & -0.2214 - \tilde{k}_{12} \\ (-0.1476e-02) - \tilde{k}_{21} & 0.8995 - \tilde{k}_{22} \end{bmatrix}$$

Solving the equations, the parameters of the feedback matrix are obtained.

$$\tilde{k}_{11} = 0.9068 - 0.9139 = -0.0071$$

$$\tilde{k}_{12} = -0.2214 - 1 = -1.2214$$

$$\tilde{k}_{21} = -(0.1476e - 02)$$

$$\tilde{k}_{22} = 0.8995 - 0.9139 = -0.0144$$

The feedback matrix is:

$$K_c = \widetilde{K_C} \cdot T^{-1} = \begin{bmatrix} -0.0071 & -1.2214 \\ -0.1476e - 02 & -0.0144 \end{bmatrix} \begin{bmatrix} 5.5545 & 5.6673 \\ -0.3704 & 0.5666 \end{bmatrix} = \begin{bmatrix} 0.4130 & -0.7323 \\ -0.2865e - 02 & -0.0165 \end{bmatrix}$$

Finally, we are going to set the Steady State through the matrix Ks:

$$x[kT + 1] = [G - H \cdot Kc]\, x[kT] + H \cdot Ks \cdot r[kT]$$

$$y[kT] = [C - D \cdot Kc]\, x[kT] + D \cdot Ks \cdot r[kT]$$

The transfer function can be calculated using:

$$tf_R(z) = C_R[z \cdot I - G_R]^{-1} \cdot H_R + D_R = [C - D \cdot Kc] \cdot [z \cdot I - G + H \cdot K_c]^{-1} \cdot H \cdot Ks + D \cdot Ks$$

In this case:

$$tf_R(z) = C \cdot [z \cdot I - G + H \cdot K_c]^{-1} \cdot H \cdot Ks$$

The position error will be null if the gain is a unit. Since the controlled system is stable:

$$gain = \lim_{z \to 1} [C \cdot [z \cdot I - G + H \cdot K_c]^{-1} \cdot H \cdot Ks] = 1$$

$$Ks = inv(C \cdot [I - G + H \cdot Kc]^{-1} \cdot H) = \begin{bmatrix} 0.8484 & -0.0788 \\ -0.0319 & 0.0488 \end{bmatrix}$$

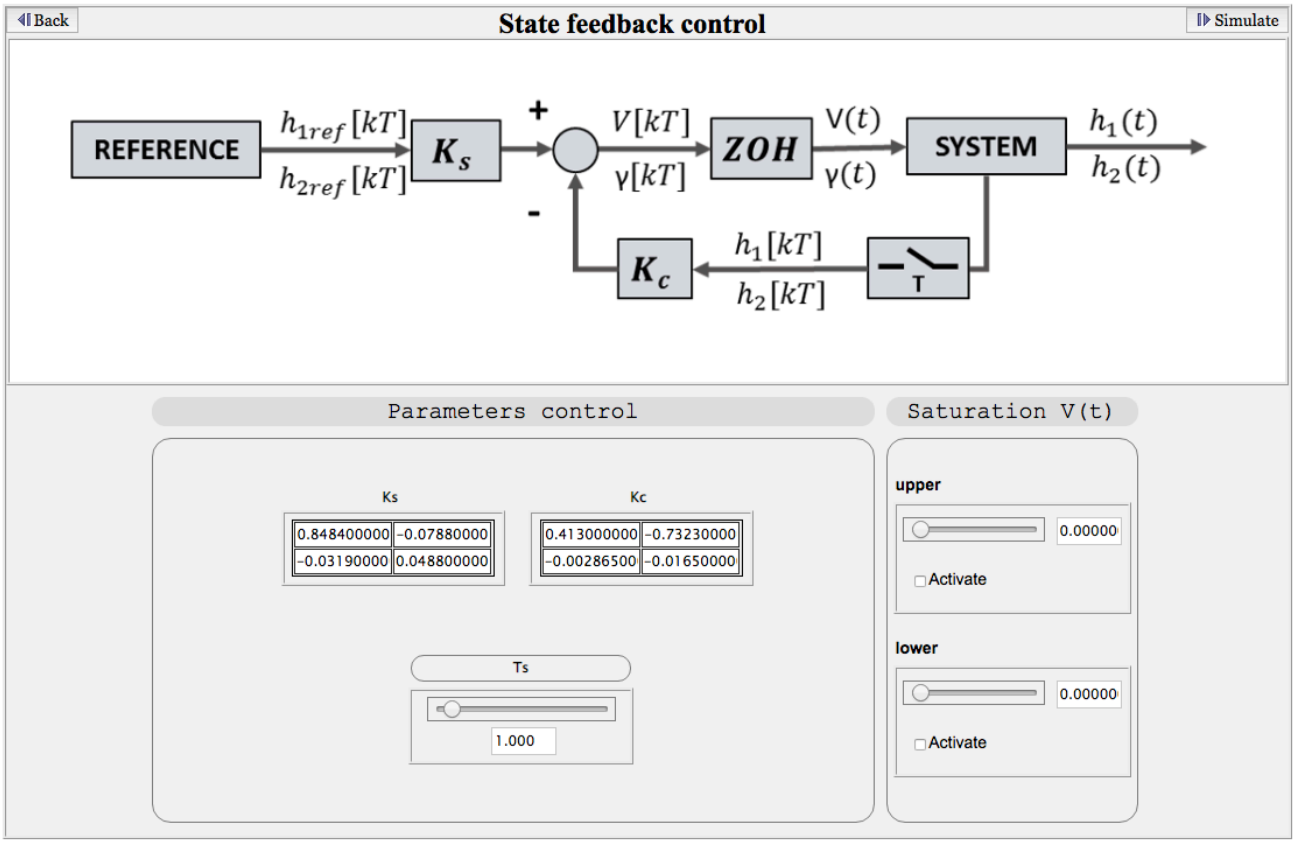


Figure 8. State feedback control window.

Exercise 3:

Open the HTML page and choose state feedback control (Figure 8). Using the parameters of the controller calculated in this section (default values), click on simulate.
Then, introduce the following inputs. Finally, check if the outputs track their respective reference values, and if the specifications are met.

Reference H1	5.5 m
Reference H2	2.5 m

Does the control system work correctly when used to the original nonlinear system?

c) OBSERVER AND STATE FEEDBACK CONTROL

The previous control requires that all the state variables are measurable. If only the input is measurable, a state observer must be added. Figure 9 shows a block diagram of the controlled system using this controller. To design this type of control, it is necessary calculate the observer matrix. To calculate it, the state space representation of the system must be in observable canonic form, so we need the transformation matrix.

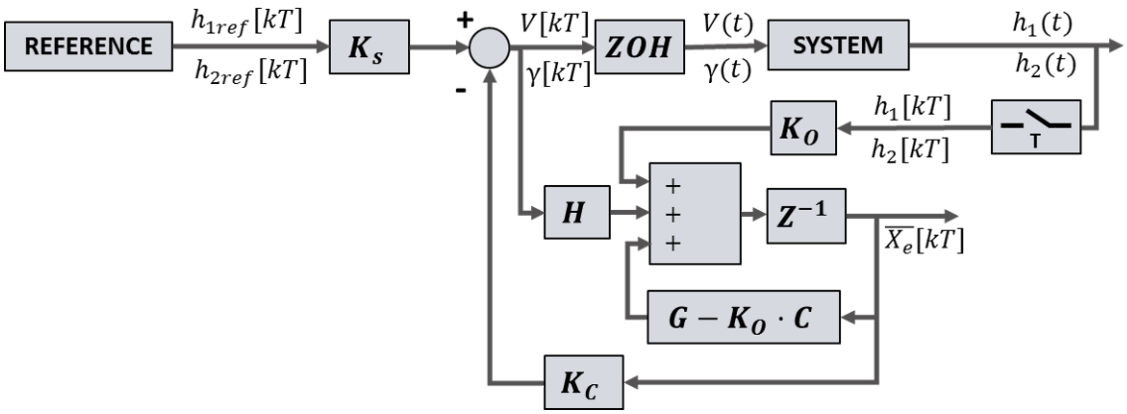


Figure 9. Control schema: Observer and state feedback.

To design the observer, the first step is to calculate the transformation matrix to the observable canonic form. It is obtained using the observability matrix. This system has two state variables ($n = 2$) and two outputs ($p = 2$).

$$P = \begin{bmatrix} c_1 \\ \vdots \\ c_p \\ c_1 \cdot G \\ \vdots \\ c_p \cdot G \\ \vdots \\ c_1 \cdot G^{n-1} \\ \vdots \\ c_p \cdot G^{n-1} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_1 \cdot G \\ c_2 \cdot G \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.9216 & 0 \\ 0 & 0.8847 \end{bmatrix}$$

where c_i is the i-th row in matrix C, which is associated to the output $y_i(t)$.

After selecting $n = 2$ linearly independent rows, these are ordered according to the output they are associated to. In the equation, n_i is the number of rows in L_o that are associated to the output y_i. The parameter d determines how many outputs take part in the feedback in this case its value is equal to the number of outputs. Besides, there is a row associated to each output so $n_1 = 1$ and $n_2 = 1$. The next step is to calculate the inverse matrix of L_o.

$$L_o = \begin{bmatrix} c_1 \\ c_1 \cdot G \\ \vdots \\ c_1 \cdot G^{n_1-1} \\ c_2 \\ c_2 \cdot G \\ \vdots \\ c_2 \cdot G^{n_2-1} \\ \vdots \\ c_d \\ c_d \cdot G \\ \vdots \\ c_d \cdot G^{n_d-1} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$L_o^{-1} = [e_1 \quad \cdots \quad e_{n_1} \quad \cdots \quad e_{n_1+n_2} \quad \cdots \quad e_{n_1+n_2+\cdots+n_d}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, we extract the final column of each block (e_{n_1}, $e_{n_1+n_2}$, $e_{n_1+n_2+\cdots+n_d}$) to obtain the transformation matrix.

$$T_o = \begin{bmatrix} e_{n_1} & \dots & G^{n_1-1} \cdot e_{n_1} & \dots & e_{n_1+\cdots+n_d} & \dots & G^{n_d-1} \cdot e_{n_1+\cdots+n_d} \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The transformation to the observable canonic form is calculated:

$$\tilde{G} = T_o^{-1} \cdot G \cdot T_o = \begin{bmatrix} 0.9216 & 0 \\ 0 & 0.8847 \end{bmatrix}$$

$$\tilde{C} = C \cdot T_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Next, we are going to design the observer matrix so that the observer estimates the state as quick as possible. In other words, the observer poles must be placed at the origin ($z = 0$). The observer characteristic polynomial is:

$$P_o(z) = z^n + \alpha_{obs_{n-1}} \cdot z^{n-1} + \cdots + \alpha_{obs_0} = z^2$$

Therefore, the desired matrix for the observer is defined in observable canonical form:

$$\tilde{G}_{obs} = \begin{bmatrix} 0 & 0 & \dots & 0 & -\alpha_{obs_0} \\ 1 & 0 & \dots & 0 & -\alpha_{obs_1} \\ 0 & 1 & \dots & 0 & -\alpha_{obs_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\alpha_{obs_{n-1}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

where α_{obs_i} is the coefficient of $P_o(z)$ associated to $z^i, i < n$. The observer matrix is obtained resolving the next equation:

$$\tilde{G}_{obs} = \tilde{G} - \widetilde{K_o} \cdot \tilde{C}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.9216 & 0 \\ 0 & 0.8847 \end{bmatrix} - \begin{bmatrix} \widetilde{k_{o_{11}}} & \widetilde{k_{o_{12}}} \\ \widetilde{k_{o_{21}}} & \widetilde{k_{o_{22}}} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.9216 - \widetilde{k_{o_{11}}} & 0 - \widetilde{k_{o_{12}}} \\ 0 - \widetilde{k_{o_{21}}} & 0.8847 - \widetilde{k_{o_{22}}} \end{bmatrix}$$

$$\widetilde{k_{o_{11}}} = 0.9216$$

$$\widetilde{k_{o_{12}}} = 0$$

$$\widetilde{k_{o_{21}}} = -1$$

$$\widetilde{k_{o_{22}}} = 0.8847$$

$$\widetilde{K_o} = \begin{bmatrix} 0.9216 & 0 \\ -1 & 0.8847 \end{bmatrix}$$

The obtained matrix is in observable canonical form, so, this matrix must be transformed to the original representation.

$$K_o = T_o \cdot \widetilde{K_o} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.9216 & 0 \\ -1 & 0.8847 \end{bmatrix} = \begin{bmatrix} 0.9216 & 0 \\ -1 & 0.8847 \end{bmatrix}$$

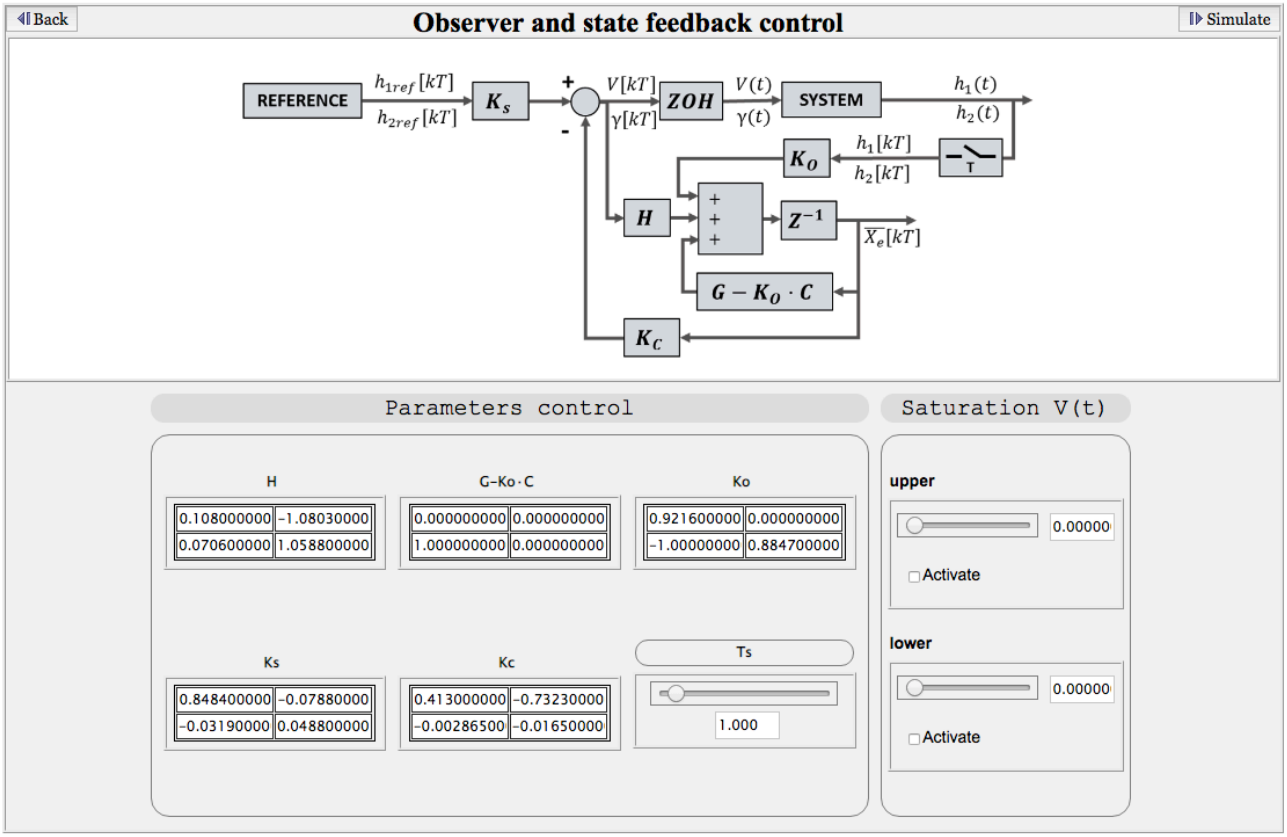


Figure 10. Observer and state feedback control window.

Exercise 4:

Open the HTML page and choose observer and state feedback control (Figure 10). Using the parameters of the controller calculated in this section (default values), click on simulate.
Then, introduces the following reference inputs. Finally, check if the outputs track their respective reference values, and if the specifications are met.

Reference H1	5.5 m
Reference H2	2.5 m

Does the control system work correctly when used to control the original nonlinear system?

d) INTEGRAL CONTROLLER, OBSERVER AND STATE FEEDBACK CONTOL

So far, we have adjusted the steady state using a gain at input, but it is not a robust solution against perturbations and errors in the model. With the purpose that the output tracks the reference input with robustness, we will feedback the output and add an integral controller. The regulator gain and the state

feedback matrix must be calculated simultaneously in this case. Figure 11 shows a block diagram of the controlled system using this controller.

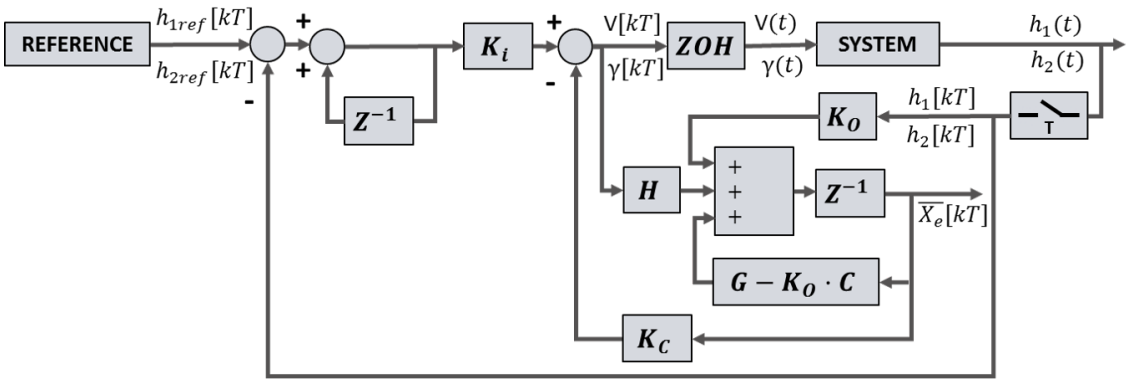


Figure 11. Control schema: Integral controller, observer and state feedback control.

First, we have to build the matrices of the new system that includes the plant and the integral controller.

$$\hat{G} = \begin{bmatrix} G & zeros(2,2) \\ -C & eye(2) \end{bmatrix} = \begin{bmatrix} 0.9216 & 0 & 0 & 0 \\ 0 & 0.8847 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\hat{H} = \begin{bmatrix} H \\ zeros(2,2) \end{bmatrix} = \begin{bmatrix} 0.1080 & -1.0803 \\ 0.0706 & 1.0588 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The order of the new system is four ($\hat{n} = n + p = 2 + 2 = 4$) and two inputs ($m = 2$), so the controllability matrix is:

$$\hat{Q} = [\hat{h}_1 \hat{h}_2 \quad \hat{G}\hat{h}_1 \hat{G}\hat{h}_2 \quad \hat{G}^2\hat{h}_1 \hat{G}^2\hat{h}_2 \quad \hat{G}^3\hat{h}_1 \hat{G}^3\hat{h}_2] =$$

$$\begin{bmatrix} 0.1080 & -1.0803 & 0.0995 & -0.9956 & 0.0917 & -0.9175 & 0.0845 & -0.8456 \\ 0.0706 & 1.0588 & 0.0625 & 0.9367 & 0.0553 & 0.8287 & 0.0489 & 0.7332 \\ 0 & 0 & -0.1080 & 1.0803 & -0.2075 & 2.0759 & -0.2993 & 2.9935 \\ 0 & 0 & -0.0706 & -1.0588 & -0.1331 & -1.9955 & -0.1883 & -2.8243 \end{bmatrix}$$

The system is controllable since the rank of the matrix Q is equal to four. The next step is to select four linearly independent columns. In this example, we select the first two columns which is associated to each input.

$$QR = [\hat{h}_1 \hat{h}_2 \quad \hat{G}\hat{h}_1 \hat{G}\hat{h}_2] = \begin{bmatrix} 0.1080 & -1.0803 & 0.0995 & -0.9956 \\ 0.0706 & 1.0588 & 0.0625 & 0.9367 \\ 0 & 0 & -0.1080 & 1.0803 \\ 0 & 0 & -0.0706 & -1.0588 \end{bmatrix}$$

The columns are ordered according to its association to each input. In this case all inputs take part in the feedback, so $d = 2$, $n_1 = 2$ and $n_2 = 2$.

$$L = [\hat{h}_1 \quad \hat{G} \cdot \hat{h}_1 \quad \dots \quad \hat{G}^{n_1 - 1} \cdot \hat{h}_2 \quad \dots \quad \hat{h}_d \quad \hat{G} \cdot \hat{h}_d \quad \dots \quad \hat{G}^{n_d - 1} \cdot \hat{h}_d]$$

$$L = [\hat{h}_1 \quad \hat{G}\cdot\hat{h}_1 \quad \hat{h}_2 \quad \hat{G}\cdot\hat{h}_2] = \begin{bmatrix} 0.1080 & 0.0996 & -1.0803 & -0.9956 \\ 0.0706 & 0.0625 & 1.0588 & 0.9367 \\ 0 & -0.1080 & 0 & 1.0803 \\ 0 & -0.0706 & 0 & -1.0588 \end{bmatrix}$$

The next step is to calculate the matrix inverse of L. Then, we extract the final row of each block to obtain the inverse of the matrix Tc.

$$L^{-1} = \begin{bmatrix} e_1^T \\ \vdots \\ e_{n1}^T \\ e_{n1+1}^T \\ \vdots \\ e_{n1+n_2}^T \\ \vdots \\ \vdots \\ e_{n1+n2+\cdots+n_d}^T \end{bmatrix} = \begin{bmatrix} e_1^T \\ e_2^T \\ e_3^T \\ e_4^T \end{bmatrix} = \begin{bmatrix} 5.5545 & 5.6673 & 5.1193 & 5.0141 \\ 0 & 0 & -5.5545 & -5.6673 \\ -0.3704 & 0.5666 & -0.3411 & 0.5014 \\ 0 & 0 & 0.3704 & -0.5666 \end{bmatrix}$$

$$T_c^{-1} = \begin{bmatrix} e_{n_1}^T \\ e_{n_1}^T \cdot \hat{G} \\ \vdots \\ e_{n1}^T \cdot \hat{G}^{n_1-1} \\ e_{n1+n_2}^T \\ \vdots \\ e_{n1+n_2}^T \cdot \hat{G}^{n_2-1} \\ \vdots \\ e_{n1+n2+\cdots+n_d}^T \\ \vdots \\ e_{n1+n2+\cdots+n_d}^T \cdot \hat{G}^{n_d-1} \end{bmatrix} = \begin{bmatrix} e_2^T \\ e_2^T \cdot \hat{G} \\ e_4^T \\ e_4^T \cdot \hat{G} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -5.5545 & -5.6673 \\ 5.5545 & 5.6673 & -5.5545 & -5.6673 \\ 0 & 0 & 0.3704 & -0.5666 \\ -0.3704 & 0.5666 & 0.3704 & -0.5666 \end{bmatrix}$$

Once the transformation matrix to the controllable canonic form is obtained, we can calculate the feedback matrix.

$$\tilde{\hat{G}} = T_c^{-1} \cdot \hat{G} \cdot T_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.9068 & 1.9068 & 0.2214 & -0.2214 \\ 0 & 0 & 0 & 1 \\ 0.1476e-02 & -(0.1476e-02) & -0.8995 & 1.8995 \end{bmatrix}$$

$$\tilde{\hat{H}} = T_c^{-1} \cdot \hat{H} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

To obtain the desired matrix we need the coefficients of the required characteristic polynomial. Considering the same specifications (critically damped with a settling time approximately equal to 52 seconds), the poles must be placed at:

$$z_{1,2} = e^{s\cdot T_s} = e^{-0.09\cdot 1} = 0.9139$$

There are only two poles, and the order of the new system is four so two more poles are required. They will be placed at $z_{3,4} = 0$.

$$P_R(z) = (z^2 - 1.8278z + 0.83521)\cdot z^2$$

Therefore, the matrix $\tilde{\hat{G}}_R$ is:

$$\tilde{\hat{G}}_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.83521 & 1.8278 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\hat{G}}_R = \tilde{\hat{G}} - \tilde{\hat{H}} \cdot \tilde{\hat{K}}_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.9068 & 1.9068 & 0.2214 & -0.2214 \\ 0 & 0 & 0 & 1 \\ 0.1476e-02 & -(0.1476e-02) & -0.8995 & 1.8995 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ \tilde{\hat{k}}_{11} & \tilde{\hat{k}}_{12} & \tilde{\hat{k}}_{13} & \tilde{\hat{k}}_{14} \\ 0 & 0 & 0 & 0 \\ \tilde{\hat{k}}_{21} & \tilde{\hat{k}}_{22} & \tilde{\hat{k}}_{23} & \tilde{\hat{k}}_{24} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.83521 & 1.8278 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.9068 - \tilde{\hat{k}}_{11} & 1.9068 - \tilde{\hat{k}}_{12} & 0.2214 - \tilde{\hat{k}}_{13} & -0.2214 - \tilde{\hat{k}}_{14} \\ 0 & 0 & 0 & 1 \\ (0.1476e-02) - \tilde{\hat{k}}_{21} & -(0.1476e-02) - \tilde{\hat{k}}_{22} & -0.8995 - \tilde{\hat{k}}_{23} & 1.8995 - \tilde{\hat{k}}_{24} \end{bmatrix}$$

so, solving the equations:

$$\tilde{\hat{k}}_{11} = -0.9068 + 0.83521 = -0.07159$$

$$\tilde{\hat{k}}_{12} = 1.9068 - 1.8278 = 0.0790$$

$$\tilde{\hat{k}}_{13} = 0.2214$$

$$\tilde{\hat{k}}_{14} = -0.2214$$

$$\tilde{\hat{k}}_{21} = 0.1476e-02$$

$$\tilde{\hat{k}}_{22} = -(0.1476e-02)$$

$$\tilde{\hat{k}}_{23} = -0.8995$$

$$\tilde{\hat{k}}_{24} = 1.8995$$

$$\tilde{\hat{K}}_R = \begin{bmatrix} -0.07159 & 0.0790 & 0.2214 & -0.2214 \\ 0.1476e-02 & -(0.1476e-02) & -0.8995 & 1.8995 \end{bmatrix}$$

The feedback matrix is obtained in controllable canonical form, for this reason we have to transform it to the original representation.

$$\hat{K}_R = \tilde{\hat{K}}_R \cdot T_c^{-1} = \begin{bmatrix} 0.5208 & 0.3228 & -0.0411 & -0.0420 \\ -0.7117 & 1.0679 & 0.3704 & -0.5668 \end{bmatrix}$$

The gain matrix of the integral controller and the feedback matrix are calculated based on the previous matrix.

$$\hat{K} = [\hat{k}_1 \quad \dots \quad \hat{k}_{\hat{n}}] = [K_c + Ki \cdot C \quad -K_i]$$

$$K_i = -[\hat{k}_{(\hat{n}+1)-p} \quad \dots \quad \hat{k}_{\hat{n}}] = \begin{bmatrix} 0.0411 & 0.0420 \\ -0.3704 & 0.5668 \end{bmatrix}$$

$$K_c = [\hat{k}_1 \quad \dots \quad \hat{k}_n] - K_i \cdot C = \begin{bmatrix} 0.4797 & 0.2808 \\ -0.3413 & 0.5011 \end{bmatrix}$$

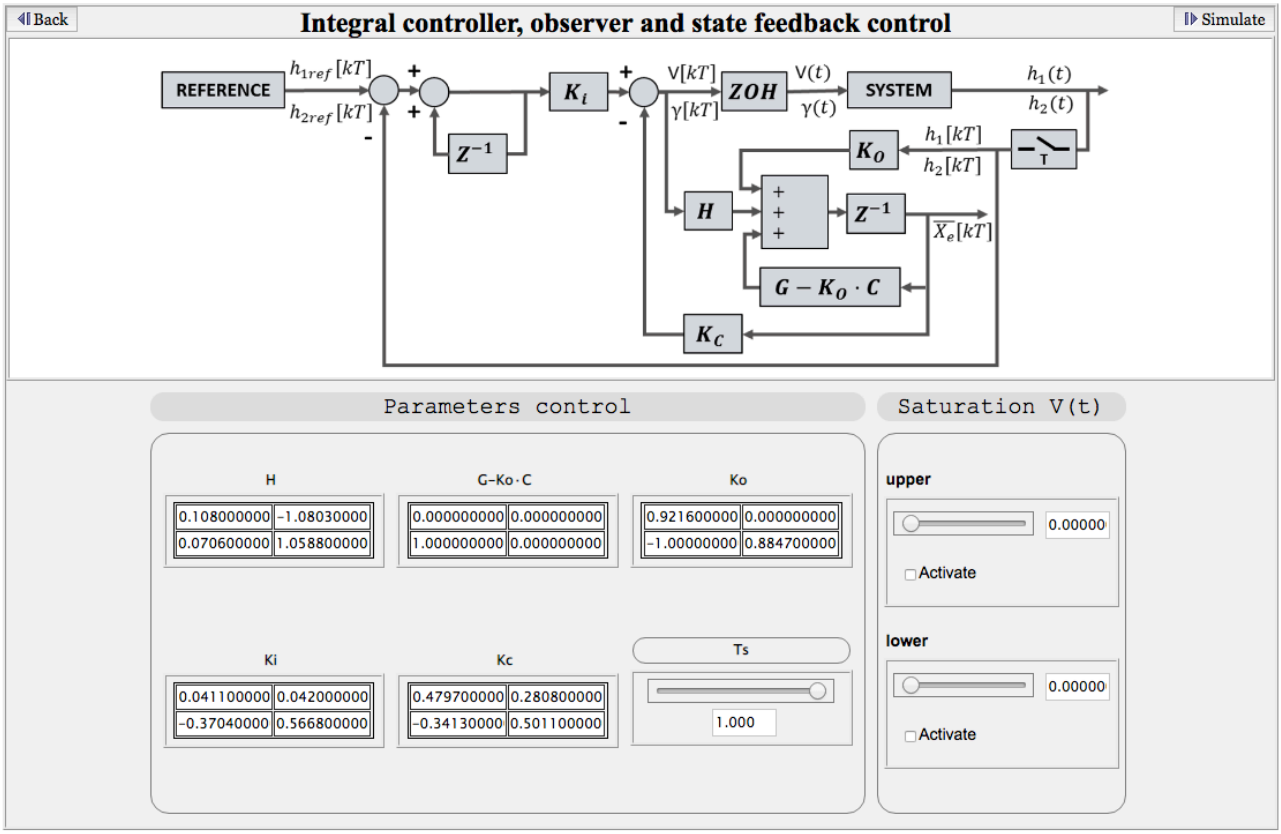


Figure 12. Integral controller, observer and state feedback control.

Exercise 5:

Open the HTML page and choose integral controller, observer and state feedback control (Figure 12). Using the parameters of the controller calculated in this section (default values), click on simulate. Then, introduces the following reference inputs. Finally, check if the outputs track their respective reference values, and if the specifications are met.

Reference H1	5.5 m
Reference H2	2.5 m

Does the control system work correctly when used to the original nonlinear system?

4. Final exercises

Exercise 6:

Starting from the scratch, introduce in the HTML page the following parameters values (international system units):

Parameters	Values	Units
Kp	3.75	$m^3/(V \cdot s)$
B1	1.5	m^2
B2	1.5	m^2
A1	40	m^2
A2	30	m^2
g	9.8	m^2/s
V_0	5	V
γ_0	0.35	

Linearized the model obtain a continuous-time state space representation and discretize it considering Ts= 0.5 sec and zero-order hold.

Exercise 7:

For the linearized model, design the following discrete-time control systems so that the dominant poles of the controlled system are placed at $s_{1,2}$=-0,2. The steady response error must be null. Then, introduce the calculated parameters into the applet and simulate. Check the evolution of the outputs when considering different reference inputs. The sample time is equal to 0.5 sec.

a) PID controller

b) State feedback control

c) Observer and state feedback control

d) Integral controller, observer and state feedback control

NOTE: if the simulation of the system appears to be slow, increase the value of SPD.

Exercise 8:

Check the behavior of the original nonlinear system when each of these controllers are used. The type of equations (linearized or original model) can be changed at the parameters window (Figure 3).